Solution Sheet 1

Note that throughout the solution the Euler formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ is used extensively. The formula will be proved in class.

1.
   - $z = \pm e^{\frac{\pi i}{4}} = \pm(\frac{\sqrt{2}+i\sqrt{2}}{2})$.
   - Write $-12 = 12e^{\pi i}$, then $z_k = \sqrt[5]{12}e^{\frac{(2k+1)\pi i}{5}}$. for $k = 0, 1, 2, 3, 4$.
   - Consider the polynomial $z^2 - z + 1 + i$. To find its roots we compute its discriminant: $\Delta = 1 - 4(1+i) = -3 - 4i$. The square root of Δ is $\pm(1-2i)$. Using the formula for quadratic equations we get that the roots are i and $1-i$.
   - This is just the quadratic equation.

2. Assume $g\colon \mathbb{C} \to \mathbb{C}$ is such an extension. Then note that $g(-1)^2 = -1$, hence $g(-1) = i$ or $g(-1) = -i$. On the other hand $g(1) = f(1) = 1$, but $g(1) = g((-1)(-1)) = g(-1)g(-1) = -1$. This is a contradiction.

3.
   - Write $\Omega = \{z \in \mathbb{C} \mid 0 < \arg(z) < \frac{\pi}{2}\}$. So if $z \in \Omega$, then $0 < \arg(z) < \pi$. Hence $f(\Omega)$ is the upper half-plane.
   - Let us parametrize the triangle:

   $$\begin{aligned} \ell_1(t) &= 1 + i + ti = 1 + i(1+t)\ ,\ t \in [0,1], \\ \ell_2(t) &= 1 + 2i + t(1-i) = 1 + t + i(2-t)\ ,\ t \in [0,1], \\ \ell_3(t) &= 1 + t + i\ ,\ t \in [0,1]. \end{aligned}$$

   In cartesian coordinates $f(z) = (x^2 - y^2) + 2ixy$. Apply f to the triangle and get the following three curves:

   $$\begin{aligned} \tilde{\ell}_1(t) &= -2t - t^2 + 2i(1+t)\ ,\ t \in [0,1], \\ \tilde{\ell}_2(t) &= -3 + 6t + 2i(2 + t - t^2)\ ,\ t \in [0,1], \\ \tilde{\ell}_3(t) &= 2t + t^2 + 2i(1+t)\ ,\ t \in [0,1]. \end{aligned}$$

   Note that the images of the vertices of the triangle are precisely the intersection between the new curves. Those are also the only points of intersection of the curves.

- The tangent vectors to the curves are given by:

$$\tilde{\ell}_1'(t) = -2 - 2t + 2i \ , \ t \in [0,1],$$
$$\tilde{\ell}_2'(t) = 6 + 2i(1-2t) \ , \ t \in [0,1],$$
$$\tilde{\ell}_3'(t) = 2 + 2t + 2i \ , \ t \in [0,1].$$

  So the tangent vector to $\tilde{\ell}_1$ at $t = 0$ is $(-2, 2)$ and at $t = 1$ is $(-4, 2)$. Similarly the tangent vectors to $\tilde{\ell}_2$ and $\tilde{\ell}_3$ are $(6, 2)$, $(6, -2)$ and $(2, 2)$, $(4, 2)$. Hence the angle between $\tilde{\ell}_1$ and $\tilde{\ell}_3$ is $\frac{\pi}{2}$. The angle between $\tilde{\ell}_1$ and $\tilde{\ell}_2$ is $\frac{\pi}{4}$ (we take the acute angle) and similarly the angle between $\tilde{\ell}_2$ and $\tilde{\ell}_3$ is $\frac{\pi}{4}$. Note that the angles are the same. This is due to the fact that the derivative of f is $2z$, therefore every curve is rotated by the same angle at each vertex.

4.
- Write $z = re^{i\theta}$, if $\bar{z} = \frac{1}{z}$, then $re^{-i\theta} = \frac{1}{r}e^{-i\theta}$, hence $r = 1$. On the other hand if $r = 1$ then $z = e^{i\theta}$ and clearly $\bar{z} = \frac{1}{z}$.
- Let $w = \frac{z}{\bar{z}}$, then $\bar{w} = \frac{\bar{z}}{z} = \frac{1}{w}$, hence by the first part of the question $|w| = 1$. As for the other direction let $w = e^{i\theta}$ some complex number on the unit circle. Fix $r > 0$ and set $z = re^{i\frac{\theta}{2}}$, then clearly $w = \frac{z}{\bar{z}}$.

5.
- Since $\frac{|a-b|}{|1-\bar{a}b|} > 0$, it suffices to show that $\frac{|a-b|^2}{|1-\bar{a}b|^2} < 1$. Note that:

$$|1-\bar{a}b|^2 - (1-|a|^2)(1-|b|^2) = (1-\bar{a}b)(1-a\bar{b}) - (1-a\bar{a})(1-b\bar{b}) =$$
$$= 1 - \bar{a}b - a\bar{b} + a\bar{a}b\bar{b} - 1 + a\bar{a} + b\ barb - a\bar{a}b\bar{b} =$$
$$= (a-b)(\bar{a}-\bar{b}) = |a-b|^2.$$

  Hence $\frac{|a-b|^2}{|1-\bar{a}b|^2} = 1 - \frac{(1-|a|^2)(1-|b|^2)}{|1-\bar{a}b|^2}$. Now since $|a| < 1$ and $|b| < 1$, we get:

$$1 - \frac{|a-b|^2}{|1-\bar{a}b|^2} = \frac{(1-|a|^2)(1-|b|^2)}{|1-\bar{a}b|^2} > 0.$$

- If either $|a| = 1$ or $|b| = 1$, then by the equality derived above we get $1 = \frac{|a-b|^2}{|1-\bar{a}b|^2}$. However, if $|a| = |b| = 1$, then it might happen that $a = \bar{b}$ and then the denominator of the fraction is 0. If we demand that $a \neq \bar{b}$, then we can allow the case when $|a| = |b| = 1$.

6. By the triangle inequality $|\sum_k \lambda_k a_k| \leq \sum_k \lambda_k |a_k|$. Now since each a_k, satisfies $0 \leq |a_k| < 1$, then $|\sum_k \lambda_k a_k| < \sum_k \lambda_k = 1$.

7. - Write $b = re^{i\theta}$, then the parametrization of the line is $z = a + re^{i\theta}t$. reparametrizing we can assume that the line is $z = a + e^{i\theta}t$. Imagine a circle of radius one around a, then θ is a choice of angle with the line through a parallel to the real axis, in which to draw our line. For a line to be perpendicular to that line, we need the angle to be $\theta + \frac{\pi}{2}$. Hence the equation of the line is $c + ibt$.
   - Recall that the complex conjugation is a reflection with respect to the real axis. Furthermore note that the transformations $z \mapsto z - a$ and $z \mapsto e^{i\theta}z$ preserve symmetry. So apply the transformation $z \mapsto e^{-i\theta}(z-a)$ to map our line to the real line. Then c is mapped to $e^{-i\theta}(c-a)$. The symmetric point with respect to the real line is $e^{i\theta}(\bar{c} - \bar{a})$. Now apply the inverse transformation $z \mapsto e^{i\theta}z + a$, to get the symmetric point $c' = e^{2i\theta}(\bar{c} - \bar{a}) + a$.

     Another way to solve this is to use the formula we've found for the line, L, perpendicular to $a + bt$ that passes through c. Since the symmetric point, c', is of the same distance from the line $a + bt$ as c, so we get that $|c - a| = |c' - a|$ (by congruence of triangles). Hence we need to find $t \in \mathbb{R}$, such that $|c - a| = |c + ibt - a|$. Let us square the equation and write $z = c - a$, then:

     $$z\bar{z} = (z + ibt)(\bar{z} - i\bar{b}t) = z\bar{z} + i\bar{z}bt - iz\bar{b}t + |b|^2t^2.$$

     Hence we get that:

     $$t(|b|^2 t - 2\,\mathrm{Im}(z\bar{b})) = 0.$$

     Hence either $t = 0$ or $t = \frac{2\,\mathrm{Im}(z\bar{b})}{|b|^2}$ ($|b| \neq 0$ or the line will collapse to a point). If $t = 0$, then the point we get is just c, so the symmetric point is the second one, i.e. $c' = c + ib\frac{2\,\mathrm{Im}(z\bar{b})}{|b|^2}$. Note that if $\mathrm{Im}(z\bar{b}) = 0$, then $(c-a)\bar{b} \in \mathbb{R}$, hence we can solve $c = a + bs$ and get:

     $$s = \frac{c-a}{b} = \frac{(c-a)\bar{b}}{|b|^2} \in \mathbb{R}.$$

     Therefore in this case c is on the line and the symmetric point is c itself.

8.
- Write $a = a_1 + ia_2$, $b = b_1 + ib_2$ and $c = c_1 + ic_2$. Denote $z = x + iy$. Hence the equation becomes:

  $$0 = (a_1 + ia_2)(x + iy) + (b_1 + ib_2)(x - iy) + c_1 + ic_2 =$$
  $$= (a_1x - a_2y + b_1x + b_2y + c_1) + i(a_1y + a_2x - b_1y + b_2x + c_2).$$

  Taking the real and imaginary part we get the following system of linear equations:

  $$\begin{pmatrix} a_1 + b_1 & b_2 - a_2 \\ a_2 + b_2 & a_1 - b_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -c_1 \\ -c_2 \end{pmatrix}.$$

  Now the determinant of the matrix is $D = a_1^2 - b_1^2 + a_2^2 - b_2^2$. If the determinant is not 0, there is a unique solution to the system. If the determinant is 0, it implies that $a_1^2 + a_2^2 = b_1^2 + b_2^2$, or in other words that $|a| = |b|$, then the solution set is either a line or an empty set. The solutions are a line if $c_2 = c_1 \frac{a_2+b_2}{a_1+b_1}$,if $a_1 + b_1 \neq 0$. If the determinant is 0 and $a_1 = -b_1$, then either $a_2 = b_2$ or $a_2 = -b_2$, so for the solution set to be a line, we need either $c_1 = 0$ or $c_2 = 0$, respectively.

- If a circle passes through two distinct points c and d. Let a be the center of the circle. Draw the radii from a to c and d. Together with the segment between c and d, the radii form an equilateral triangle. Hence if we draw the height from a to the segment connecting c and d, it also divides the segment in two. Therefore a lies on the line perpendicular to the segment between c and d through its center.

  A more analytic way is to take the line perpendicular to the segment connecting c and d through its center. Then every point on this line is of equal distance from c and from d (draw the triangles and check that they are congruent). Let a be the center of the circle. Denote $|c - a| = r = |d - a|$. Recall that the line that passes through the points c and d is given by the parametrization $d + (c - d)t$ and the middle of the segment connecting those points is $\frac{c+d}{2}$. So the line perpindicular to the segment through its center is given by $\frac{c+d}{2} + i(c - d)t$ (previous exercise). In order to show that a is on that line we need to find a **real** t, such that $a = \frac{c+d}{2} + i(c - d)t$. Clearly we have:

  $$t = \frac{2a - c - d}{2i(c - d)}.$$

However, this number need not be real in general. WE will show that this number is in fact real. Denote $z_1 = a - c$ and $z_2 = a - d$. Then:

$$t = \frac{2a - c - d}{2i(c - d)} = \frac{z_1 + z_2}{2i(z_2 - z_1)} = \frac{(z_1 + z_2)(\bar{z}_2 - \bar{z}_1)}{2i|z_2 - z_1|^2} =$$
$$= \frac{z_1\bar{z}_2 - |z_1|^2 + |z_2|^2 - z_2\bar{z}_1}{2i|z_2 - z_1|^2}.$$

Now recall that $|z_1| = |z_2| = r$. Hence $t = \frac{\mathrm{Im}(z_1\bar{z}_2)}{|z_2 - z_1|^2} \in \mathbb{R}$.

9.
- Note that $\omega = e^{\frac{i\pi}{n}}$ and therefore $\omega^n = 1$. Now if h is not a multiple of n, then $\sum_{k=0}^{n-1} \omega^{kh} = \frac{1-\omega^{nh}}{1-\omega^h}$. The denominator is not 0 because of our assumption. However, $\omega^{nh} = 1$ and therefore the sum is 0.
- If h is a multiple of n,this sum is either 0 if n is even or 1 if n is odd. Now assume that h is not a multiple of n and write:

$$\sum_{k=0}^{n-1} (-\omega^h)^k = \frac{1 - (-\omega^h)^n}{1 + \omega^h} = \frac{1 - (-1)^n}{1 + e^{\frac{2\pi i h}{n}}}$$

Hence if n is even this sum is 0 and if n is odd this sum is:

$$\sum_{k=0}^{n-1} (-\omega^h)^k = \frac{2}{1 + e^{\frac{2\pi i h}{n}}} = \frac{2(1 + e^{-\frac{2\pi i h}{n}})}{2 + 2\cos(\frac{2\pi h}{n})} = 1 - i\frac{\sin(\frac{2\pi h}{n})}{1 + \cos(\frac{2\pi h}{n})}.$$